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ON A CLASS OF EXACT SOLUTIONS OF A NON-AXISYMMETRIC CONTACT PROBLEM FOR AN INHOMOGENEOUS ELASTIC HALF-SPACE*

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A non-axisymmetric mixed boundary-value problem is considered concerning the pressure (in the absence of friction and adhesion forces) of a stiff circular-planform stamp with a base of aribitrary shape on an inhomogeneous elastic half-space. The shear modulus of the half-space material is constant while Poisson's ratio is an arbitrary piecewise-continuous function of the depth. By using the theory of dual integral equations associated with the generalized Hankel integral operator, the problem is reduced to a sequence of one-dimensional Fredholm integral equations of the second kind.

It is shown that the integral equations obtained allow exact solutions to be constructed for periodic law of variation of the half-space material elastic properties with depth. The solution of a non-axisymmetric problem regarding the eccentric impression of a stamp with a flat base is presented as a example, on the basis of which the influence of inhomogeneity of the elastic material on the magnitude of the stamp displacement parameters is investigated. An asymptotic analysis is performed for the solution in the case when the elastic characteristics of the material become rapidly oscillating functions.

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Some problems for inhomogeneous materials with variable Poisson's ratio where investigated earlier /1-4/.

1. We consider the problem of the pressure of a stiff stamp on an inhomogeneous elastic half-space $R_+^3 = \{\mathbf{x}^\circ\colon x_3 > 0\}$, where $\mathbf{x}^\circ = (x_1, x_2, x_3)$ is a point in space R^3. The shear modulus of the half-space material is constant ($\mu = \mathrm{const} > 0$) while Poisson's ratio $\nu = \nu(x_3)$ is an arbitrary function satisfying the standard conditions $-1 < \nu(x_3) \leqslant 1/2$ /5/. The elastic modulus of the material $E = 2\mu(1+\nu)$ is here a positive function of the depth.

The vector equilibrium equation in displacements for the model of elastic material inhomogeneity under consideration has the form (no bulk forces)

$$\Delta \mathbf{u}(\mathbf{x}^\circ) + \nabla[\eta(x_3)\nabla\cdot\mathbf{u}(\mathbf{x}^\circ)] = 0, \quad \mathbf{x}^\circ \in R_+^3$$
$$\mathbf{u} = (u_1, u_2, u_3), \quad \eta = (1-2\nu)^{-1} \tag{1.1}$$

where Δ and ∇ are, respectively, the Laplace operator and the gradient in R^3.

We will write the boundary conditions and conditions at infinity

$$\sigma_{\alpha 3}(\mathbf{x}, 0) = 0$$
$$\sigma_{33}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \notin \Omega \tag{1.2}$$
$$u_3(\mathbf{x}, 0) = \delta + \varepsilon_1 x_1 + \varepsilon_2 x_2 - f(\mathbf{x}), \quad \mathbf{x} \in \Omega$$
$$u_i(\mathbf{x}^\circ) \to 0, \quad \sigma_{ij}(\mathbf{x}^\circ) \to 0 \quad \text{as} \quad |\mathbf{x}^\circ| \to \infty \tag{1.3}$$

as well as equilibrium equations for the stamp

$$P = -\iint_\Omega \sigma_{33}(\mathbf{x}, 0)\, d\mathbf{x} \tag{1.4}$$
$$M_1 = -\iint_\Omega \sigma_{33}(\mathbf{x}, 0)\, x_2\, d\mathbf{x}, \quad M_2 = -\iint_\Omega \sigma_{33}(\mathbf{x}, 0)\, x_1\, d\mathbf{x}$$

Here $\mathbf{x} = (x_1, x_2)$, Ω is the contact area, σ_{ij} are stress tensor components, δ, ε_1, ε_2 are previously unknown displacement parameters for the stamp as a solid body, $f(\mathbf{x})$ is a given function describing the shape of the stamp base, and P, M_1 and M_2 are the principal vector and the principal moments of the forces applied to the stamp. In (1.2) and (1.3) and everywhere later the subscripts i, j take the values 1, 2, 3, and the subscript α only 1 and 2. Summation is not performed over repeated subscripts.

The general solution of (1.1) has the form

$$\mathbf{u} = \mathbf{B} + \nabla b, \quad \mathbf{B} = (B_1, B_2, B_3) \tag{1.5}$$
$$\Delta\mathbf{B} = 0, \quad \Delta b = -{}^1/_2\gamma\nabla\cdot\mathbf{B}, \quad \gamma = [1-\nu(x_3)]^{-1}$$

In the case of a homogeneous material (1.5) reduce to the Freiberger solution /5/, which is a modification of the well-known Papkovich-Neuber representation. The representation (1.5) remains valid even in the more general case when $\nu = \nu(\mathbf{x}^\circ)$. Other forms of the general solution of (1.1) are indicated in /4, 6/.

The stress tensor components are expressed in terms of the functions B and b by the relationships (δ_{ij} is the Kronecker delta, and the comma before the subscript denotes differentiation with respect to the appropriate variable x_i)

$$\mu^{-1}\sigma_{ij} = \delta_{ij}\nu\gamma\nabla\cdot\mathbf{B} + B_{i,j} + B_{j,i} + 2b_{,ij} \tag{1.6}$$

Without loss of generality, it is possible to set $B_1 = B_2 = 0$ when there are no shear stresses (see (1.2)) on the half-space boundary. As a result of such a simplification, relationships (1.5) and (1.6) take the form

$$u_\alpha = b_{,\alpha}, \quad u_3 = B + b_{,3} \tag{1.7}$$
$$\Delta B = 0, \quad \Delta b = -{}^1/_2\gamma B_{,3} \tag{1.8}$$
$$\mu^{-1}\sigma_{12} = 2b_{,12}, \quad \mu^{-1}\sigma_{\alpha 3} = (B + 2b_{,3})_{,\alpha} \tag{1.9}$$
$$\mu^{-1}\sigma_{\alpha\alpha} = \nu\gamma B_{,3} + 2b_{,\alpha\alpha}, \quad \mu^{-1}\sigma_{33} = [(2-\nu)\gamma B + 2b_{,3}]_{,3}$$

where the notation $B = B_3$ is used.

2. We introduce the two-dimensional Fourier integral transform operator in the variables x_1 and x_2

$$F\{\varphi(\mathbf{x}^\circ)\}(\mathbf{k}, x_3) \equiv \varphi^F(\mathbf{k}, x_3) = \iint_{-\infty}^{\infty} \varphi(\mathbf{x}^\circ)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}, \quad \mathbf{k} = (k_1, k_2) \tag{2.1}$$

Applying it to relationship (1.8) and solving the ordinary differential equations obtained here, taking the conditions (1.3) at infinity into account, we find ($C(\mathbf{k})$ and $D(\mathbf{k})$ are arbitrary functions)

$$
\begin{gathered}
B^F(\mathbf{k}, x_3) = C(\mathbf{k})\, e^{-kx_3} \\
b^F(\mathbf{k}, x_3) = [D(\mathbf{k}) - {}^1/_4 C(\mathbf{k})\, m(x_3)]\, e^{-kx_3} - {}^1/_4 C(\mathbf{k})\, l(k, x_3)\, e^{kx_3} \\
m(x_3) = \int_0^{x_3} \gamma(x_3)\, dx_3, \quad l(k, x_3) = \int_{x_3}^{\infty} \gamma(x_3)\, e^{-2kx_3}\, dx_3, \quad k = |\mathbf{k}|
\end{gathered} \tag{2.2}
$$

Applying the Fourier transform to (1.7) and (1.9) and substituting (2.2) into the relationships obtained, in particular, we obtain

$$
\begin{gathered}
u_3^F(\mathbf{k}, x_3) = [C(\mathbf{k}) - kD(\mathbf{k}) + {}^1/_4 kC(\mathbf{k})\, m(x_3)]\, e^{-kx_3} - \\
{}^1/_4 kC(\mathbf{k})\, l(k, x_3)\, e^{kx_3} \\
\sigma_{\alpha 3}^F(\mathbf{k}, x_3) = i\mu k_\alpha \Lambda^+(\mathbf{k}, x_3) \\
\sigma_{33}^F(\mathbf{k}, x_3) = -\mu k \Lambda^-(\mathbf{k}, x_3) \\
\Lambda^\pm(\mathbf{k}, x_3) = {}^1/_2 kC(\mathbf{k})\, l(k, x_3)\, e^{kx_3} \pm \\
[2kD(\mathbf{k}) - C(\mathbf{k}) - {}^1/_2 kC(\mathbf{k})\, m(x_3)]\, e^{-kx_3}
\end{gathered} \tag{2.3}
$$

The first boundary condition (1.2) will be satisfied if we set $4kD(\mathbf{k}) = [2 - kl(k, 0)]\, C(\mathbf{k})$. Here (2.3) take the following form on the boundary of the half space R_+^3

$$
\begin{gathered}
u_3^F(\mathbf{k}, 0) = {}^1/_2 C(\mathbf{k}), \quad \sigma_{\alpha 3}^F(\mathbf{k}, 0) = 0 \\
\sigma_{33}^F(\mathbf{k}, 0) = -\mu k^2 L(k)\, C(\mathbf{k}), \quad L(k) = l(k, 0)
\end{gathered} \tag{2.4}
$$

Eliminating the function C from the first and third relationships in (2.4), we set up a connection between the Fourier transforms of the normal stresses and the normal displacements at points of the boundary of the half-space R_+^3

$$
\sigma_{33}^F(\mathbf{k}, 0) = -2\mu k^2 L(k)\, u_3^F(\mathbf{k}, 0) \tag{2.5}
$$

We introduce polar coordinates in the plane $x_3 = 0$ by the relationships $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ and we consider the case when the contact area has the shape of a circle of radius a: $\Omega = \{r, \varphi\colon 0 \leqslant r \leqslant a,\ -\pi < \varphi \leqslant \pi\}$. We will represent the functions $\sigma(\mathbf{x}) = -\sigma_{33}(\mathbf{x}, 0)$ and $u(\mathbf{x}) = u_3(\mathbf{x}, 0)$ by complex Fourier series

$$
\begin{gathered}
\begin{Bmatrix} u(r, \varphi) \\ \sigma(r, \varphi) \end{Bmatrix} = \sum_{n=-\infty}^{\infty} \begin{Bmatrix} u_n(r) \\ \sigma_n(r) \end{Bmatrix} e^{in\varphi} \\
\begin{Bmatrix} u_n(r) \\ \sigma_n(r) \end{Bmatrix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \begin{Bmatrix} u(r, \varphi) \\ \sigma(r, \varphi) \end{Bmatrix} e^{-in\varphi}\, d\varphi
\end{gathered} \tag{2.6}
$$

after which we obtain a connection between the functions $u_n(r)$ and $\sigma_n(r)$ by changing to polar coordinates in (2.5) and integrating with respect to the angular coordinate (the notation for the generalized Hankel integral operator /7/ is used)

$$
\begin{gathered}
S_{N/2,0}\{u_n(r)\}(\rho) = [2\mu\rho^2 L(\rho)]^{-1} S_{N/2,0}\{\sigma_n(r)\}(\rho), \quad N = |n| \\
\left(S_{\nu,\beta}\{\varphi(r)\}(\rho) = \left(\frac{2}{\rho}\right)^{\beta} \int_0^{\infty} r^{1-\beta} \varphi(r)\, J_{2\nu+\beta}(\rho r)\, dr\right)
\end{gathered} \tag{2.7}
$$

Inverting the relationship (2.7), we obtain

$$
\begin{gathered}
u_n(r) = (2\mu)^{-1}(1 - \nu_0)\, r S_{(N-1)/2,1}\{(1 + G)\, \Psi_n\}(r) \\
\Psi_n(\rho) = S_{N/2,0}\{\sigma_n(r)\}(\rho), \quad \nu_0 = \nu(0) \\
1 + G(\rho) = [2(1 - \nu_0)\rho L(\rho)]^{-1}, \quad \lim_{\rho\to\infty} G(\rho) = 0
\end{gathered} \tag{2.8}
$$

where obviously,

$$
\sigma_n(r) = S_{N/2,0}\{\Psi_n(\rho)\}(r) \tag{2.9}
$$

Changing to polar coordinates in the second and third boundary conditions of (1.2) and substituting the representation (2.6) therein, taking relationships (2.8) and (2.9) into acount, we arrive at a system of dual integral equations in the auxiliary functions $\Psi_n(\rho)$

$$S_{(N-1)/2,1}\{[1+G(\rho)]\Psi_n(\rho)\}(r)=f_n^*(r),\quad 0\leqslant r<a \tag{2.10}$$
$$S_{N/2,0}\{\Psi_n(\rho)\}(r)=0,\quad a<r<\infty$$
$$f_n^*(r)=2\mu[(1-\nu_0)r]^{-1}f_n(r)$$
$$f_n(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\delta+\varepsilon_1 r\cos\varphi+\varepsilon_2 r\sin\varphi-f(r\cos\varphi,r\sin\varphi)]e^{-in\varphi}d\varphi$$

Substituting the representation

$$\Psi_n(\rho)=2\mu\pi^{-1/2}(1-\nu_0)^{-1}S_{N/2,-1/2}\{x^{-1}g_n(x)\}(\rho) \tag{2.11}$$

into (2.10), we find that $g_n(x)\equiv 0$ for $a<x$ while the functions $g_n(x)$ satisfy a Fredholm integral equation of the second kind

$$g_n(x)+x^{1/2}\int_0^a g_n(s)s^{1/2}ds\int_0^\infty G(t)J_{N-1/2}(xt)J_{N-1/2}(st)t\,dt= \tag{2.12}$$
$$\frac{1}{x^N}\frac{d}{dx}\int_0^x\frac{f_n(r)r^{N+1}dr}{(x^2-r^2)^{1/2}}$$

in the segement $0\leqslant x\leqslant a$.

The functions $\sigma_n(r)$ governing the contact pressure distribution under the stamp are expressed in terms of the auxiliary functions $g_n(x)$ by using the quadratures

$$\sigma_n(r)=-\frac{2\mu r^{N-1}}{\pi(1-\nu_0)}\frac{d}{dr}\int_r^a\frac{g_n(x)x^{1-N}dx}{(x^2-r^2)^{1/2}},\quad 0\leqslant r<a \tag{2.13}$$

Introducing the complex principal moment $M=M_2+iM_1$ and changing to polar coordinates in the stamp equilibrium Eqs.(1.4), we obtain

$$P=2\pi\int_0^a\sigma_0(r)r\,dr=4\mu(1-\nu_0)^{-1}\int_0^a g_0(x)dx \tag{2.14}$$
$$M=2\pi\int_0^a\sigma_{-1}(r)r^2dr=8\mu(1-\nu_0)^{-1}\int_0^a g_{-1}(x)x\,dx$$

Therefore, the solution of the contact problem for a circular stamp with arbitrary base shape reduces to constructing an infinite sequence of functions $g_n(x)$ satisfying Fredholm integral equations of the second kind (2.12) and the stamp equilibrium Eqs.(2.14). The mentioned sequence ordinarily contains just several non-zero terms for stamp base shapes of practical interest.

We will later limit ourselves to considering the non-axisymmetric problem of the eccentric impression of a circular stamp with a flat base. In this case $f(\mathbf{x})=0$ and

$$f_0(r)=\delta,\ f_{\pm1}(r)={}^1/_2(\varepsilon_1\mp i\varepsilon_2)r \tag{2.15}$$
$$f_n(r)=0 \text{ for } |n|>1$$

Substituting the last relationship of (2.15) into (2.12) we conclude that $g_n(x)=0$ for $|n|>1$ so that in this case only the functions $g_0(x)$ and $g_{\pm1}(x)$ are non-zero.

Changing to new auxiliary functions

$$y_0(x)=\delta^{-1}g_0(x),\ y_{\pm1}(x)=(\varepsilon_1\mp i\varepsilon_2)^{-1}g_{\pm1}(x) \tag{2.16}$$

in the remaining integral equations of (2.12), we obtain

$$y_0(x)+\int_0^a K_0(x,s)y_0(s)ds=1,\quad 0\leqslant x\leqslant a \tag{2.17}$$
$$y_{\pm1}(x)+\int_0^a K_{\pm1}(x,s)y_{\pm1}(s)ds=x,\quad 0\leqslant x\leqslant a$$
$$\left\{\begin{matrix}K_0(x,s)\\K_{\pm1}(x,s)\end{matrix}\right\}=\frac{2}{\pi}\int_0^\infty G(t)\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}xt\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\}st\,dt$$

from which it follows, in particular, that the functions $y_{\pm 1}(x)$ are equal to one another, unlike $g_{\pm 1}(x)$.

Substituting (2.16) into (2.14), we arrive at the following formulas connecting the stamp displacement parameters with the external load

$$\delta = 2\theta Y_0^{-1}P,\ \varepsilon_1 = \theta Y_1^{-1}M_2,\ \varepsilon_2 = \theta Y_1^{-1}M_1 \tag{2.18}$$

$$\theta = (8\mu)^{-1}(1-\nu_0),\quad Y_0 = \int_0^a y_0(x)\,dx,\quad Y_1 = \int_0^a y_1(x)\,x\,dx$$

The contact pressure under the stamp takes the following form in the case under consideration:

$$\sigma(r,\varphi) = -\frac{1}{2\pi}\left\{\frac{P}{Y_0 r}\frac{d}{dr}\int_r^a \frac{y_0(x)\,x\,dx}{(x^2-r^2)^{1/2}} + \right. \tag{2.19}$$

$$\left.\frac{M_2\cos\varphi + M_1\sin\varphi}{Y_1}\frac{d}{dr}\int_r^a \frac{y_1(x)\,dx}{(x^2-r^2)^{1/2}}\right\},\quad 0 \leqslant r < a$$

3. The known exact solutions of spatial contact problems for an inhomogeneous half-space /8-12/ refer to the case when the elastic characteristics of the half-space material are monotonic functions of the depth. The inhomogeneity model under consideration in this paper enables us to obtain an analytic solution of the contact problem for a periodic law of variation of the elastic characteristics.

For example, let

$$\gamma(x_3) = b_1 + b_2\cos b_3 x_3 \tag{3.1}$$

so that Poisson's ratio and the elastic modulus are periodic functions of the depth with period $T = 2\pi b_3^{-1}$. Here

$$G(t) = \frac{b\lambda^2}{t^2+\lambda^2},\quad b = \frac{b_2}{b_1},\quad \lambda^2 = \frac{\beta^2}{1+b},\quad \beta^2 = \frac{b_3^2}{4} \tag{3.2}$$

and the kernels of the integral Eqs.(2.17) are evaluated in elementary functions, where

$$K_0(x,s) = b\lambda\begin{cases} e^{-\lambda x}\operatorname{ch}\lambda s, & x \geqslant s \geqslant 0\\ e^{-\lambda s}\operatorname{ch}\lambda x, & s \geqslant x \geqslant 0\end{cases} \tag{3.3}$$

and the expression for $K_1(x,s)$ corresponds to the replacement of cosh by sinh in (3.3).

Taking account of (3.3) we write the integral equation in the function $y_0(x)$ as

$$y_0(x) = 1 - b\lambda e^{-\lambda x}\int_0^x y_0(s)\operatorname{ch}\lambda s\,ds - b\lambda\operatorname{ch}\lambda x\int_x^a y_0(s)\,e^{-\lambda s}\,ds \tag{3.4}$$

and in particular

$$y_0(0) = 1 - b\lambda\int_0^a y_0(s)\,e^{-\lambda s}\,ds \tag{3.5}$$

Differentiating (3.4) with respect to x we obtain

$$y_0'(0) = 0 \tag{3.6}$$

Differentiating (3.4) twice with respect to x, we find that the solution of the integral Eq.(3.4) satisfies a second-order ordinary differential equation

$$y_0''(x) - \beta^2 y_0(x) = -\lambda^2 \tag{3.7}$$

as well as the initial conditions (3.5) and (3.6), as has been established.

The solution of the Cauchy problem (3.5)-(3.7) has the form

$$y_0(x) = \frac{1}{1+b}\left(1 + \frac{b\lambda\operatorname{ch}\beta x}{\lambda\operatorname{ch}\beta a + \beta\operatorname{sh}\beta a}\right) \tag{3.8}$$

In an analogous manner we establish that the function $y_1(x)$, and therefore, also $y_{-1}(x)$, are determined by the formula

$$y_1(x) = \frac{1}{1+b}\left[x + \frac{b(1+\lambda a)\,\mathrm{sh}\,\beta x}{\lambda\,\mathrm{sh}\,\beta a + \beta\,\mathrm{ch}\,\beta a}\right] \tag{3.9}$$

Substituting relationships (3.8) and (3.9) into (2.18), we obtain the following representations for the stamp displacement parameters:

$$\delta = \delta^\circ[1 + \zeta(b, \varkappa)],\ \varepsilon_\alpha = \varepsilon_\alpha{}^\circ[1 + \xi(b, \varkappa)],\ \alpha = 1, 2 \tag{3.10}$$

$$\delta^\circ = \frac{(1-\nu_0)P}{4\mu a}, \quad \varepsilon_1{}^\circ = \frac{3(1-\nu_0)M_2}{8\mu a^3}, \quad \varepsilon_2{}^\circ = \frac{3(1-\nu_0)M_1}{8\mu a^3}$$

$$\zeta = b\frac{\varkappa + (\varkappa g - 1)\,\mathrm{th}\,\varkappa}{\varkappa + (\varkappa g + b)\,\mathrm{th}\,\varkappa}$$

$$\xi = b\frac{\varkappa(\varkappa^2 g - 3h) + (\varkappa^3 + 3h)\,\mathrm{th}\,\varkappa}{\varkappa(\varkappa^2 g + 3bh) + (\varkappa^3 - 3bh)\,\mathrm{th}\,\varkappa}$$

$$\varkappa = \beta a = {}^1/_2 a b_3, \quad g = (1+b)^{1/2}, \quad h = \varkappa + g$$

where δ° and $\varepsilon_\alpha{}^\circ$ are the stamp displacement parameters for a homogeneous half-space with Poisson's ratio $\nu = \nu_0$. Therefore, the quantities ζ and ξ, which depend on the two dimensionless parameters b and $\varkappa$ characterize the influence of the elastic material inhomogeneity on the stamp displacement parameters. It can be seen that $\zeta = \xi = 0$ for $b = 0$ or for $\varkappa = 0$ (these values correspond to a homogeneous material).

Let us set up the domains of allowable values of the parameters b and $\varkappa$. Since the function $\cos b_3 x_3$ in (3.1) is even, without loss of generality we can set $b_3 \geqslant 0$. Therefore, the domain of allowable values of the parameter $\varkappa$ is the semi-axis $[0, +\infty)$. The range of variation of the parameter b is determined by the interval of allowable values of the function $\nu(x_3)$. In particular, if $0 \leqslant \nu(x_3) \leqslant {}^1/_2$ (these constraints hold for the majority of natural and structural materials), then

$$(2\nu_0 - 1)(3 - 2\nu_0)^{-1} \leqslant b \leqslant \nu_0(2 - \nu_0)^{-1}$$

In this case the domain of allowable values of the parameter b is the segment $[-{}^1/_3, {}^1/_3]$. In general, the range of variation of the parameter b can be broader since materials with negative value of Poisson's ratio do exist /13/.

It is of particular interest to investigate the behaviour of the stamp displacement parameters as $\varkappa \to \infty$ (which is equivalent to the case $b_3 \to \infty$ for a fixed value of a) when the functions $\nu(x_3)$ and $E(x_3)$ become rapidly oscillating and their period $T \to 0$. Making the mentioned passage to the limit in (3.10), we find

$$\lim_{\varkappa\to\infty} \delta = \delta^\circ(1+b), \quad \lim_{\varkappa\to\infty} \varepsilon_\alpha = \varepsilon_\alpha{}^\circ(1+b) \tag{3.11}$$

where the relations $\zeta(b, \infty) = \xi(b, \infty) = b$ were taken into account.

We introduce into the consideration a homogeneous elastic half-space with constant shear modulus μ and Poisson's ratio ν^* by denoting the appropriate values of the stamp displacement parameters by δ^* and $\varepsilon_\alpha{}^*$ (these values are obtained from the expression for δ° and $\varepsilon_\alpha{}^\circ$ by replacing ν_0 by ν^*). If we set $\nu^* = \nu_0 - b(1 - \nu_0)$, then on the basis of (3.11) we obtain

$$\lim_{\varkappa\to\infty} \delta = \delta^*, \quad \lim_{\varkappa\to\infty} \varepsilon_\alpha = \varepsilon_\alpha{}^* \tag{3.12}$$

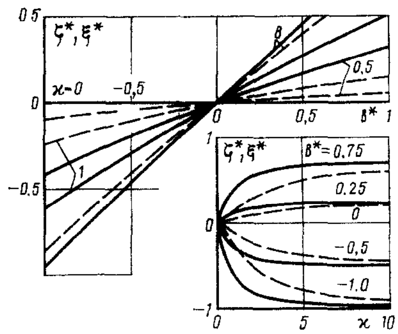
Therefore, as the parameter $\varkappa$ increases, the stamp displacements on an inhomogeneous half-space with Poisson's ratio (3.1) tend to the corresponding values for a homogeneous half-space with Poisson's ratio ν^*. The quantity ν^* can be expressed as follows in terms of the mean value of the function $\gamma(x_3)$ in the period

$$\nu^* = 1 - \frac{1}{\langle\gamma(x_3)\rangle}, \quad \langle\gamma(x_3)\rangle = \frac{1}{T}\int_0^T \gamma(x_3)\,dx_3$$

from which it is seen, in particular, that $\nu^* \in [0, {}^1/_2]$ for $0 \leqslant \nu(x_3) \leqslant {}^1/_2$.

The figure shows graphs of $\zeta^* = 3\zeta$ (solid lines) and $\xi^* = 3\xi$ (dashes) against $\varkappa$ for fixed values of $b^* = 3b$ and against b^* for fixed values of $\varkappa$. We note that $|\zeta|$ and $|\xi|$ increase monotonically as $\varkappa$ increases and approach the limit value $|b|$ quite rapidly. The graphs of ζ and ξ against b are almost linear: strictly linear relations are obtained for $\varkappa = \infty$ when $\zeta = \xi = b$.

4. In conclusion, we mention a more general class of periodic models of elastic material inhomogeneity for which it is possible to obtain analytic solutions of the non-axisymmetric contact problem under consideration.



Let (as before, $\mu = \mathrm{const}$)

$$\gamma(x_3) = \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos\frac{k\pi x_3}{H_k} \tag{4.1}$$

where a_k and H_k are certain constants. For $m = 1$ the law (3.1) is obtained from (4.1) as a special case (apart from the notation). If (4.1) is considered as a segment of a Fourier cosine series governing the function $\gamma(x_3)$ in the interval $[0, H]$ (in this case $H_k = H$ for $k = 1, 2, \ldots, m$), then

$$a_k = \frac{2}{H}\int_0^H \gamma(x_3)\cos\frac{k\pi x_3}{H}\,dx_3$$

For the law (4.1)

$$G(t) = \sum_{k=1}^{m} \frac{G_k}{t^2 + g_k^2}$$

where the constants G_k and g_k are expressed in terms of a_k and H_k so that the kernel of the integral Eqs.(2.17) are represented by sums (containing m terms) of kernels of the type (3.3). The integral equations obtained here reduce to ordinary differential equations of order $2m$ that enable us to construct analytic solutions. The calculations associated with the realization of the approach mentioned become more awkward and are not presented here but a description of the method for solving equations of this kind can be found in /14, 15/.

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CONTACT PROBLEMS FOR SYSTEMS OF ELASTIC HALF-PLANES*

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Static and stationary dynamic problems are considered for systems of N elastic isotropic half-planes attached by arbitrary sections of their boundaries. Outside the attached sections the half-planes are contiguous to stamps and flexible facings. The kinds of mixed boundary conditions, whose number can reach six, are given in each half-plane independently. In particular, the most important case of a plane $N=2$ does not require the presence of specular symmetry of the types on opposite edges of slits, which provides the possibility of studying new classes of problems of the cutting, wedging, and debonding of inclusions.

The procedure proposed for the solution enables the problems in question to be reduced, in a general formulation, to Hilbert-Riemann bounary-value problems on N-sheeted Riemann surfaces defined by bifurcation and the law of attachement of the sheets. If the problem of constructing the algebraic function of the Riemann surface obtained along its bifurcation is solved (for $N=2$, i.e., for a hyperelliptic surface, this function is well-known for $N \leqslant 4$ and it can obviously also be found in the general case), then the corresponding contact problem is solved by quadratures. Examples are considered.

1. Let $\{R_k\}_1^N$ be a set of specimens of the complex plane $z = x + iy$; $S_k = \{z \in R_k: y > 0\}$, $k = 1, 2, \ldots, N^+$, $N^+ < N$, is the upper elastic half-plane $S_k = \{z \in R_k: y < 0\}$, $k = N^+ + 1, N^+ + 2, \ldots, N$ is the lower elastic half-plane, and Γ_k is the boundary of S_k. Each k-th upper half-plane is contiguous to $N_k \in [1, N^-]$, $N^- = N - N^+$, by some lower half-planes, $\Gamma_{kl}' \subset \Gamma_k$ and $\Gamma_{lk}' \subset \Gamma_l$ are contact boundaries of the domains S_k and S_l and coincide when R_k and R_l are superimposed $\Gamma_{kl}' \cap \Gamma_{km}' = \varnothing$ for $l \neq m$. Let the elastic domain $S_1 \cup S_2 \cup \ldots S_N$ "be connected" in the sense that for any k and $l \in [1, N]$ a chain of boundaries $\Gamma_{kr}', \Gamma_{rs}', \ldots, \Gamma_{qp}', \Gamma_{pl}'$ connecting the domains S_k and S_l exists: Γ_k' is the union of Γ_{kl}' over all N_k values of l, $\Gamma_k'' = \Gamma_k \setminus \Gamma_k'$, μ is the shear modulus, ν is Poisson's ratio, and ρ is the density of the material S. In the general case different kinds of fundamental or mixed boundary conditions P_k for different k are formulated on Γ_k and the nature of the singularities allowable at the separation points is indicated; stress field intensities satisfying the equilibrium and connectedness conditions of the domain S are given at infinity in each half-plane. The boundaries Γ_k move with the identical constant subsonic velocity $c \geqslant 0$ relative to the fixed domains S_k. It is required to determine the elastic deformations of the domain S.

In the case $N=2$ certain fundamental problems for a homogeneous plane with slits are solved by quadratures for $P_1 = P_2$ and $P_1 \neq P_2$ /1/, for a composite plane and for $P_1 = P_2$ /2/; mixed problems are solved in the same form just for $P_1 = P_2$ /3-6/, i.e., for specularly symmetric kinds of conditions on opposite edges of the slits.

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